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Masafumi Hayashi

University of the Ryukyus, Department of Mathematical Sci- ences, Faculty of Science, Nishihara-cho, Okinawa 903-0213, Japan, hayashi@math.u-ryukyu.ac.jp

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REGULARITY OF THE LOCAL TIME OF DIFFUSIONS ON THE POSITIVE REAL LINE WITH REFLECTION AT ZERO

MASAFUMI HAYASHI*

Abstract. We study the joint law of $(X_t(x), L_t(x))$ where $X_t(x)$ is the solution of a one dimensional stochastic differential equation on $(0, +\infty)$ with reflection at zero, and $L_t(x)$ is its local time. In particular, we give some representation formula of the distribution of $(X_t(x), L_t(x))$, and we investigate the regularity of the joint density with respect to the local time argument under ellipticity and mild regularity conditions on coefficients of $X_t(x)$.

1. Introduction

Let $T > 0$ be fixed. We consider the following stochastic differential equation (SDE) on $D := (0, \infty)$ with reflecting boundary conditions:

$$X_t(x) = x + \int_0^t \sigma(X_s(x))\, dW_s + \int_0^t b(X_s(x))\, ds + L_t(x), \qquad 0 \le t \le T. \quad (1.1)$$

Here $x \in \bar{D} := [0, \infty)$ and $\{W_t\}_{0\le t\le T}$ is a one dimensional standard Brownian motion on the canonical filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$. We assume that $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration generated by W. We say that $\{(X_t(x), L_t(x))\}_{0\le t\le T}$ is the solution to (1.1) if it satisfies

L1. Both $X_t(x)$ and $L_t(x)$ are non-negative, continuous and $\mathcal{F}_t$-adapted processes satisfying (1.1);

L2. $L_0(x) = 0$ and $t \to L_t(x)$ is increasing P-a.s.;

L3. The measure $dL_s(x)$ is carried by $\partial D := \{0\}$:

$$L_t(x) = \int_0^t \mathbf{1}_{\partial_D}(X_s(x))\, dL_s(x).$$

Diffusions with reflecting boundary condition such as (1.1) appear naturally in various applications. In the 80s, the existence and uniqueness of the solution to SDEs with reflecting boundary conditions were studied by many authors, see [9], [11] and references therein. In recent years, several aspects of diffusions with reflecting boundary condition has been studied. For example Wong–Zakai approximation for diffusions with reflecting boundary condition, which describes a simple relation between the solution of SDE and that of ordinary differential equations, are studied, see [6] and [1]. Deuschel and Zambotti [5] proved the solution to SDE's

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* Corresponding author.

with reflecting boundary conditions are pathwise differentiable with respect to the initial value, and they obtained Bismut-Elworthy's formula for the gradient of the transition semigroup $E[f(X_t(x))]$, see also [3].

Tsuchiya [12] obtained, by using parametrix methodology, an approximation for $E[f(X_t(x))]$ and investigated the existence of the density of $X_t(x)$, see also [2]. The main purpose of the present paper is to generalize the methodology exposed by [12] and to give some approximation formula of the transition semigroup $P_tf(x,\ell) := E[f(X_t(x),\ell+L_t(x))]$ under mild regularity conditions on the coefficients of $X_t(x)$. The main difference between an approximation for $E[f(X_t(x))]$ and that for $P_tf(x,\ell)$ arises from the singularity of the distribution of the local time. Indeed the local time process $L_t(x)$ stays zero until $X_t(x)$ touches zero, hence the joint law of $(X_t(x),\ell+L_t(x))$ is not absolutely continuous with respect to the Lebesgue measure. For this reason, an approximation has two parts, that is, approximations which do not touch the boundary, and approximations which touch the boundary. This combination of effects will happen in any iteration of the procedure of the approximation and will generate combinations of approximations with two parts. This creates difficulties in the analysis. We will show in Thoerem 2.4 that the joint law of $(X_t(x),\ell+L_t(x))$ has the following form:

$$P(X_t(x)\in dx',\,\ell+L_t(x)\in d\ell')=p_t(x;x')\,dx'\,\delta_\ell(d\ell')+p_t(x,\ell;x',\ell')1_A\,dx'\,d\ell'. \tag{1.2}$$

Here we denote by $\delta_\ell(d\ell')$ the Dirac point mass concentrated at ℓ, and put

$$A:=\{(x,\ell,x',\ell')\in\mathbb{R}^4:\,x,\,x'\in D,\,\ell'>\ell\}. \tag{1.3}$$

The first term in the right hand side in (1.2) have a point mass at the current value of local time and corresponds to the case where the reflected process does not touch the boundary. The second term is absolutely continuous with respect to the Lebesgue measure and corresponds to the case where the reflected process touches the boundary. In Theorem 2.4, Gaussian upper estimates for $p_t(x;x')$ and $p_t(x,\ell;x',\ell')$ are also obtained.

As an application of Theorem 2.4 we study the regularity of $p_t(x,\ell;x',\ell')$ with respect to ℓ'. The regularity of the transition density of the solution to SDE is usually related to the regularity of its coefficients. Even in the present case, we can also show, by following a similar approach as in [4], the differentiability of $p_t(x;x')$ and $p_t(x,\ell;x',\ell')$ with respect to x. This application also enables us to see the differentiability of $x\to E[g(X_t(x)):T_0(x)\ge t]$ or $x\to E[f(X_t(x),\ell+L_t(x)):T_0<t]$, where g and f are bounded measurable functions, and $T_0(x)$ is the first hitting time to zero by $X_t(x)$. On the other hand, it seems that the regularity of $p_t(x,\ell;x',\ell')$ with respect to local time argument ℓ' has not been studied enough. As Nualart and Vives [10] showed, a Brownian local time belongs to some fractional order Sobolev space in the sense of Malliavin calculus. Hence one cannot apply standard techniques from Malliavin calculus to the study of regularity of local time. However it is known that the distribution of the Brownian local time is smooth except for the boundary. In the present paper, as an application of Theorem 2.4, we will investigate the regularity of $p_t(x,\ell;x',\ell')$ with respect to ℓ' under mild regularity conditions on the coefficients of (1.1). In particular we

will show that, even though we do not assume any differentiability of coefficients of (1.1), $p_t(x, \ell; x', \ell')$ is smooth with respect to ℓ' except for the boundary $\ell' = \ell$ (Theorem 2.5).

The organization of the paper is as follows: assumptions and main results are exhibited in Section 2. We also mention some auxiliary results that we frequently use in the present paper. We give an approximation for the semigroup $P_t f(x, \ell)$ in Section 3. Based on the approximation for $P_t f(x, \ell)$, we prove the representation formula (1.2) in Section 4. In Section 5 we investigate the regularity of $p_t(x, \ell; x', \ell')$ with respect to ℓ'. In Appendices, we prove auxiliary results and the first step formula that is a key formula in the theory of parametrix methodology.

Notations: We denote by $C_b(\bar{D} \times \mathbb{R})$ the space of continuous bounded functions on $\bar{D} \times \mathbb{R}$. The sup-norm of the function f will be denoted by $\|f\|_\infty$. We denote by $\mathbb{N}_0$ the set of all non-negative integers. To describe the joint law of the approximation, we will use the following notation for Hermite type functions

$$\tilde{H}_n(x, a) = \left(\frac{d}{dx}\right)^n \left[\frac{1}{\sqrt{2\pi a}} \exp(-\frac{x^2}{2a})\right], \qquad n \in \mathbb{N}_0,$$

as well as $H_n(x, a) = \tilde{H}_n(x, a)\tilde{H}_0(x, a)^{-1}$. $B(s, t)$ and $\Gamma(z)$ denote Beta and Gamma functions respectively. We frequently use the following formula:

$$\int_0^t s^{\beta-1} (t-s)^{\gamma-1} \, ds = B(\beta, \gamma) \, t^{\beta+\gamma-1} = \frac{\Gamma(\beta) \, \Gamma(\gamma)}{\Gamma(\beta+\gamma)} \, t^{\beta+\gamma-1}. \tag{1.4}$$

We also use the Mittag-Leffler function:

$$E_{a,b}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}, \qquad z \in \mathbb{R}. \tag{1.5}$$

We remark that the sum converges for any $z \in \mathbb{R}$.

As usual constants are denoted by the letters C and M, and it may change value from one line to the next. These constant may depend on T, and other constants appearing in the assumptions.

2. Assumptions and Main Results

2.1. Assumptions. Throughout the present paper, we assume that the coefficients of SDE (1.1) satisfy the following conditions:

Assumption (H)

H1. $a := \sigma^2$ is uniformly elliptic and bounded measurable:

$$0 < \underline{a} := \inf_{x \in \bar{D}} a(x) \leq \overline{a} := \sup_{x \in \bar{D}} a(x) < \infty.$$

H2. a is α-Hölder continuous for some $\alpha \in (1/2, 1)$:

$$\|a\|_\alpha := \sup_{x, y \in \bar{D}, \, x \neq y} \frac{|a(x) - a(y)|}{|x - y|^\alpha} < \infty.$$

H3. $b : \bar{D} \to \mathbb{R}$ is bounded and measurable.

Remark 2.1. (1) For standard results on the existence and uniqueness of solutions for reflected SDEs and their properties we refer the reader to e.g. Lions-Sznitman [9] and Tsuchiya [12].

(2) Note that a is α-Hölder continuous for any $\alpha \in (1/2, 1)$, if a is Lipschitz continuous. Hence our main results (Theorems 2.3–2.5) mentioned below are also applicable to the case $\alpha = 1$.

2.2. The approximation. Let $x, z \in \bar{D}$ be fixed. In order to construct an approximation for $P_t f(x, \ell)$, we consider the following Skorokhod equation:

$$\bar{X}_t^{(z)}(x) = x + \sigma(z) W_t + \bar{L}_t^{(z)}(x). \tag{2.1}$$

We say that $(\bar{X}_t^{(z)}(x), \bar{L}_t^{(z)}(x))$ is the solution to (2.1) if $(\bar{X}_t^{(z)}(x), \bar{L}_t^{(z)}(x))$ satisfies the following conditions:

- $\bar{\mathbf{L}}\mathbf{1}$. Both $\bar{X}_t^{(z)}(x)$ and $\bar{L}_t^{(z)}(x)$ are non-negative, continuous and $\mathcal{F}_t$-adapted processes satisfying (2.1);
- $\bar{\mathbf{L}}\mathbf{2}$. $\bar{L}_0^{(z)}(x) = 0$ and $t \to \bar{L}_t^{(z)}(x)$ is increasing P-a.s.;
- $\bar{\mathbf{L}}\mathbf{3}$. The measure $d\bar{L}_s^{(z)}(x)$ is carried by ∂D:

$$\bar{L}_t^{(z)}(x) = \int_0^t \mathbf{1}_{\partial D}(\bar{X}_s^{(z)}(x))\, d\bar{L}_s^{(z)}(x).$$

There exists a unique pathwise solution to (2.1), see e.g., Lemma 6.14 in Chap. 3 of [8]. The following proposition can be deduced from Proposition 8.1 in Chap. 2 of [8]. Recall that A is defined by (1.3).

Proposition 2.2. *The joint law of* $(\bar{X}_t^{(z)}(x), \ell + \bar{L}_t^{(z)}(x))$ *is given by*

$$P(\,(\bar{X}_t^{(z)}(x), \ell + \bar{L}_t^{(z)}(x)) \in dx'\, d\ell') = \bar{\pi}_t^{(z)}(x; x') dx'\, \delta_\ell(d\ell') + \bar{\bar{\pi}}_t^{(z)}(x, \ell; x', \ell') dx'\, d\ell',$$

where we have used the functions

$$\bar{\pi}_t^{(z)}(x; x') := \left(\tilde{H}_0(x - x', a(z)t) - \tilde{H}_0(x + x', a(z)t)\right) 1_{\bar{D} \times \bar{D}}(x, x'),$$

and

$$\bar{\bar{\pi}}_t^{(z)}(x, \ell; x', \ell') := -2\tilde{H}_1(x + x' + \ell' - \ell, a(z)t)\, 1_{\bar{A}}(x, \ell, x', \ell').$$

Here $\bar{A}$ *stands for the closure of* A.

2.3. Main results. We approximate $P_t f(x, \ell)$ by using the following operator

$$\mathcal{P}_t f(x, \ell) = \int_D f(x', \ell) \bar{\pi}_t^{(x')}(x, x')\, dx' + \int_\ell^\infty \int_D f(x', \ell') \bar{\bar{\pi}}_t^{(x')}(x, \ell; x', \ell')\, dx'\, d\ell'.$$

This operator may look strange at first sight but one may interpret it as a "reversed" transition operator. To evaluate the remainder of the approximation, we use the infinitesimal generators associated to $(X_t(x), L_t(x))$ and $(\bar{X}_t^{(z)}(x), \bar{L}_t^{(z)}(x))$:

$$\mathcal{L} f(x, \ell) = \mathcal{L}_x f(x, \ell) := b(x) \partial_x f(x, \ell) + \frac{1}{2} a(x) \partial_x^2 f(x, \ell),$$

$$\bar{\mathcal{L}}^{(z)} f(x, \ell) = \bar{\mathcal{L}}_x^{(z)} f(x, \ell) := \frac{1}{2} a(z) \partial_x^2 f(x, \ell),$$

respectively. Both of the domains of these operators contain the following set

$$\mathcal{D} := \left\{ f \in C_b(\bar{D} \times \mathbb{R}) \, ; \quad \begin{array}{l} (\partial_x f(x,\ell), \partial_x^2 f(x,\ell), \partial_\ell f(0,\ell)) \text{ exists and} \\ \text{is continuous, bounded and satisfies} \\ \partial_x f(0,\ell) + \partial_\ell f(0,\ell) = 0, \qquad (\ell \in \mathbb{R}) \end{array} \right\}. \tag{2.2}$$

Define

$$S_t f(x,\ell) = \int_D f(x',\ell)\, \kappa_t(x;x')\, dx' + \int_\ell^\infty \int_D f(x',\ell') \kappa_t(x,\ell;x',\ell') \; dx' \, d\ell',$$

where we have used the functions

$$\kappa_t(x;x') := \left(\mathcal{L}_x - \bar{\mathcal{L}}_x^{(x')}\right) \bar{\pi}_t^{(x')}(x;x') 1_{\bar{D}\times\bar{D}}(x,x'),$$
$$\kappa_t(x,\ell;x',\ell') := \left(\mathcal{L}_x - \bar{\mathcal{L}}_x^{(x')}\right) \bar{\pi}_t^{(x')}(x,\ell;x',\ell')\, 1_{\bar{A}}(x,\ell,x',\ell').$$

Theorem 2.3. *Suppose that Assumption (H) holds. For $f \in C_b(\bar{D} \times \mathbb{R})$, we define*

$$I_t^0(f)(x,\ell) = \mathcal{P}_t f(x,\ell),$$

and for $n \geq 1$

$$I_t^n(f)(x,\ell) = \int_0^t du_1 \int_0^{u_1} \; du_2 \cdots \int_0^{u_{n-1}} \; du_n \mathcal{P}_{u_n} S_{u_{n-1}-u_n} \cdots S_{u_1-u_2} S_{t-u_1} f(x,\ell).$$

Then, we have

$$P_t f(x,\ell) = \sum_{n=0}^\infty I_t^n(f)(x,\ell),$$

where the sum in the right hand side converges uniformly in $(x,\ell) \in \bar{D} \times \mathbb{R}$.

By using Theorem 2.3 one can show representation formula (1.2). In order to construct approximations for $p(x;x')$ and $p_t(x,\ell;x',\ell')$, define recursively

$$\begin{cases} p_t^0(x;x') = & \bar{\pi}_t^{(x')}(x;x') 1_{\bar{D}\times\bar{D}}(x,x'), \\ p_t^n(x;x') = & \left(\displaystyle\int_0^t \int_D p_s^{n-1}(x;\xi) \kappa_{t-s}(\xi;x')\, d\xi\, ds \right) 1_{\bar{D}\times\bar{D}}(x,x'), \qquad \text{for } n \geq 1. \end{cases}$$

For $n \in \mathbb{N}$, put

$$q_t^n(x,\ell;x',\ell') = \left(\int_0^t \int_D p_s^{n-1}(x;\xi) \kappa_{t-s}(\xi,\ell;x',\ell')\, d\xi\, ds \right) 1_{\bar{A}}(x,\ell,x',\ell').$$

Then we define recursively

$$\begin{cases} p_t^0(x,\ell;x',\ell') = & \bar{\pi}_t^{(x')}(x,\ell;x',\ell'), \\ p_t^n(x,\ell;x',\ell') = & q_t^n(x,\ell;x',\ell') + \displaystyle\int_0^t S_{t-s}^* p_s^{n-1}(x,\ell;\cdot,\cdot)(x',\ell')\, ds, \;\; \text{for } n \geq 1, \end{cases}$$

for $(x,\ell,x',\ell') \in \bar{A}$, and set $p_t^n(x,\ell;x',\ell') = 0$ for $(x,\ell,x',\ell') \in \bar{A}^c$. Here, S_t^* is the adjoint operator of S_t. In particular, for $(x,\ell,x',\ell') \in A$, we can write

$$\int_0^t S_{t-s}^* p_s^{n-1}(x,\ell;\cdot,\cdot)(x',\ell')\,ds = \int_0^t \int_D p_s^{n-1}(x,\ell;\xi,\ell')\ \kappa_{t-s}(\xi;x')\,d\xi\,ds$$
$$+ \int_0^t \int_\ell^{\ell'} \int_D p_s^{n-1}(x,\ell;\xi,\lambda)\kappa_{t-s}(\xi,\lambda;x',\ell')\ d\xi\,d\lambda\,ds.$$

To simplify the notation we denote

$$h_t^\mu(x,\ell;x',\ell') = \tilde{H}_0(x+x'+\ell'-\ell,\mu\bar{a}t). \tag{2.3}$$

Here $\mu > 1$ and $\bar{a}$ is the constant introduced in **H1** in Assumption (H). Recall that we denote by $E_{a,b}(z)$ the Mittag-Leffler function, see (1.5).

Theorem 2.4. *Suppose that Assumption (H) holds. For any $\mu > 1$ there exist positive constants C and M such that the following assertions hold true:*

1. *For each $t \in (0,T]$, $p_t(x;x') := \sum_{n=0}^\infty p_t^n(x;x')$ converges uniformly in $(x,x') \in \bar{D}^2$. Moreover we have that, for any $(t,x,x') \in (0,T]\times\bar{D}\times\bar{D}$,*
$$p_t(x;x') \le CE_{\alpha/2,1}(Mt^{\alpha/2})\tilde{H}_0(x-x',\mu\bar{a}t).$$
2. *For each $t \in (0,T]$, $p_t(x,\ell;x',\ell') := \sum_{n=0}^\infty p_t^n(x,\ell;x',\ell')$ converges uniformly in $(x,\ell,x',\ell') \in \bar{A}$. Moreover we have that, for any $(t,x,\ell,x',\ell') \in (0,T]\times\bar{A}$,*
$$p_t(x,\ell;x',\ell') \le CE_{\alpha/2,1/2}(Mt^{\alpha/2})\ t^{-1/2}\,h_t^\mu(x,\ell;x',\ell').$$
3. *Formula (1.2) holds true.*

As an application of Theorem 2.4, we will show the smoothness of $p_t(x,\ell;x',\ell')$ with respect to ℓ'. For $\delta \in (0,1)$, we put

$$A_\delta = \{(x,\ell,x',\ell') \in \mathbb{R}^4 :\ x,\ x' \in D,\ \ell'-\ell > \delta\}. \tag{2.4}$$

Theorem 2.5. *Suppose that Assumption (H) holds. Then $p_t(x,\ell;x',\ell') : A \to \mathbb{R}$ is infinitely differentiable with respect to ℓ', and for any $(t,x,\ell,x',\ell') \in (0,T]\times\bar{A}_\delta$*

$$\left|\partial_{\ell'}^m p_t(x,\ell;x',\ell')\right| \le \frac{C\,E_{\alpha/2,1/2}(Mt^{\alpha/2})}{\delta^m}\ t^{-1/2}h_t^\mu(x,\ell;x',\ell'),$$

holds true. Both constants C and M depend on m, μ, α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$, and T, but are independent of δ.

Remark 2.6. One can see easily see that $p_t^n(x,\ell;x',\ell') = p_t^n(x,0;x',\ell'-\ell)$ for any n and also $p_t(x,\ell;x',\ell') = p_t(x,0;x',\ell'-\ell)$. Thus Theorem 2.5 also assures the infinite differentiability with respect to ℓ.

2.4. Some auxiliary results. Although, in the present paper, constants may change from line to line, we sometimes need to fix constants when we use inequalities (2.5)–(2.8) below.

Proposition 2.7. *For any $\mu > 1$, there exists a positive constant M_μ such that*

(i) *For $n = 0, 1, 2$, for any $z \in \bar{D}$ and for any $(t, x, x') \in (0, T] \times \bar{D}^2$,*

$$\left|\partial_{x'}^n \pi_t^{(z)}(x; x')\right| \le M_\mu t^{-n/2} \left(1 \wedge \frac{x'}{t^{1/2}}\right) \tilde{H}_0(x - x', \mu \bar{a} t). \tag{2.5}$$

(ii) *For any $z \in \bar{D}$ and for any $(t, x, \ell, x', \ell') \in (0, T] \times \bar{A}$,*

$$\left|\pi_t^{(z)}(x, \ell; x', \ell')\right| \le M_\mu t^{-1/2} h_t^\mu(x, \ell; x', \ell'). \tag{2.6}$$

(iii) *For any $(t, x, x') \in (0, T] \times \bar{D}^2$,*

$$\left|\kappa_t(x; x')\right| \le M_\mu t^{(\alpha-2)/2} \left(1 \wedge \frac{x'}{t^{1/2}}\right) \tilde{H}_0(x - x', \mu \bar{a} t). \tag{2.7}$$

(iv) *For any $(t, x, \ell, x', \ell') \in (0, T] \times \bar{A}$,*

$$\left|\kappa_t(x, \ell; x', \ell')\right| \le M_\mu t^{(\alpha-3)/2} h_t^\mu(x, \ell; x', \ell'). \tag{2.8}$$

The constant M_μ depends not only on μ, but also on α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$ and T.

Remark 2.8. The proof of Proposition 2.7 will be given in Appendices. In [7], the authors studied a similar estimate to (2.5) see Lemma 5.3 in [7].

We will also frequently use the following inequalities.

Proposition 2.9. *Let $\mu > 1$ be fixed.*

(i) *For any $0 < s < t < T$, and for any $x, x' \in \bar{D}$ we have*

$$\int_D \tilde{H}_0(x - \xi, \mu \bar{a} s)\, \tilde{H}_0(\xi - x', \mu \bar{a}(t - s))\, d\xi \le \tilde{H}_0(x - x', \mu \bar{a} t). \tag{2.9}$$

(ii) *For any $0 < s < t < T$, and for any $(x, \ell, x', \ell') \in \bar{A}$ we have*

$$\int_D \tilde{H}_0(x - \xi, \mu \bar{a} s)\, h_{t-s}^\mu(\xi, \ell; x', \ell')\, d\xi \le h_t^\mu(x, \ell; x', \ell'). \tag{2.10}$$

(iii) *For any $0 < s < t < T$, and for any $(x, \ell, x', \ell') \in \bar{A}$ we have*

$$\int_\ell^{\ell'} \int_D \left|h_s^\mu(x, \ell; \xi, \lambda)\, \kappa_{t-s}(\xi, \lambda; x', \ell')\right| d\xi\, d\lambda \le M_\mu (t - s)^{\alpha/2 - 1}\, h_t^\mu(x, \ell; x', \ell'). \tag{2.11}$$

Proof. We omit the proof of (2.9) and (2.10) because they are similar to the proof of (2.11). Let $1 < \mu' < \mu$ be fixed. Put $p = \mu/\mu'$ and $q = (1 - 1/p)^{-1}$. Then note that p, $q > 1$ and $(1/p) + (1/q) = 1$. We have that for any $\xi, x' \in D$ and $\ell \le \lambda \le \ell'$,

$$\begin{aligned} h_{t-s}^{\mu'}(\xi, \lambda; x', \ell') &= (2\pi \bar{a} \mu' p q (t - s))^{1/2} h_{t-s}^{\mu' p}(\xi, \lambda; x', \ell')\, h_{t-s}^{\mu' q}(\xi, \lambda, x', \ell') \\ &\le (2\pi \bar{a} \mu' p q (t - s))^{1/2} h_{t-s}^{\mu}(\xi, \lambda; x', \ell') \tilde{H}_0(\ell' - \lambda, \mu' \bar{a} q (t - s)). \end{aligned} \tag{2.12}$$

Here the last inequality follows from the fact that $\xi + x' + \ell' - \lambda \ge \ell' - \lambda \ge 0$. By using this and (2.8) with $\mu = \mu'$, we have

$$|\kappa_{t-s}(\xi, \lambda; x', \ell')| \le M_\mu (t - s)^{\alpha/2 - 1} h_{t-s}^\mu(\xi, \lambda; x', \ell') \tilde{H}_0(\ell' - \lambda, \mu' \bar{a} q (t - s)).$$

Here we replace the constant $M_{\mu'}(2\pi\mu' pq)^{1/2}$ by M_μ because we may assume that $M_{\mu'}(2\pi\mu' pq)^{1/2} \le M_\mu$ holds. Thus we obtain

$$\begin{aligned}
&\int_\ell^{\ell'} \int_D |h_s^\mu(x,\ell;\xi,\lambda)\,\kappa_{t-s}(\xi,\lambda;x',\ell')| d\xi d\lambda \\
&\le M_\mu (t-s)^{\alpha/2-1} \int_\ell^{\ell'} \tilde{H}_0(\ell'-\lambda, \mu'\bar{a}q(t-s)) d\lambda \int_D h_s^\mu(x,\ell;\xi,\lambda)\, h_{t-s}^\mu(\xi,\lambda;x',\ell') d\xi \\
&\le M_\mu (t-s)^{\alpha/2-1} \int_{\mathbb{R}} \tilde{H}_0(\ell'-\lambda, \mu'\bar{a}q(t-s)) d\lambda \times h_t^\mu(x,\ell;x',\ell')
\end{aligned}$$

Here, in the last inequality, we use the semigroup property of $\tilde{H}_0(x, \mu\bar{a}t)$. Therefore (2.11) holds true. □

3. The Proof of Theorem 2.3

The following lemma plays a fundamental role in the theory of parametrix methodology. We will prove Lemma 3.1 in Appendices.

Lemma 3.1. *Assume the Hypotheses (H) hold. For any $f \in C_b(\bar{D} \times \mathbb{R})$, we have*

$$P_t f(x,\ell) - \mathcal{P}_t f(x,\ell) = \int_0^t P_u S_{t-u} f(x,\ell)\, du. \tag{3.1}$$

Recall that $h_t^\mu(x,\ell;x',\ell')$ is defined by (2.3).

Proposition 3.2. *There exists $C > 0$ such that for any $f(x,\ell) \in C_b(\bar{D} \times \mathbb{R})$*

$$\|\mathcal{P}_t f\|_\infty \le C\|f\|_\infty,$$

and

$$\|S_t f\|_\infty \le C t^{(\alpha-2)/2} \|f\|_\infty,$$

hold for any $t \in (0,T]$. Here C depends on α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$ and T.

Proof. For any $(x,\ell,x',\ell') \in \bar{A}$, note that $(x+x')(\ell'-\ell) \ge 0$. Hence one can see that

$$h_t^2(x,\ell;x',\ell') \le (4\pi\bar{a}T)^{1/2}\, \tilde{H}_0(x+x', 2\bar{a}t) \tilde{H}_0(\ell'-\ell; 2\bar{a}t). \tag{3.2}$$

Using this and Proposition 2.7 with $\mu = 2$, we can obtain the desired estimates. □

3.1. Proof of Theorem 2.3. By using Lemma 3.1, we have

$$\begin{aligned}
P_t f =& \mathcal{P}_t f + \int_0^t P_{u_1} S_{t-u_1} f\, du_1 \\
=& I_t^0 f + \int_0^t \left(\mathcal{P}_{u_1} S_{t-u_1} f + \int_0^{u_1} P_{u_2} S_{u_1-u_2} S_{t-u_1} f\, du_2 \right) du_1 \\
=& I_t^0 f + I_t^1 f + \int_0^t \int_0^{u_1} P_{u_2} S_{u_1-u_2} S_{t-u_1} f\, du_2\, du_1.
\end{aligned}$$

By iterating this procedure N times, we obtain $P_t f = \sum_{n=0}^N I_t^n f + R_{N+1}$, with

$$R_{N+1} = \int_0^t du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_N} du_{N+1} P_{u_{N+1}} S_{u_N - u_{N+1}} \cdots S_{t-u_1} f.$$

Hence, using Proposition 3.2 we obtain

$$\begin{aligned}\|R_{N+1}\|_\infty &\le \int_0^t du_1 \cdots \int_0^{u_N} du_{N+1} \prod_{j=1}^{N+1} \left(C(u_{j-1}-u_j)^{(\alpha-2)/2}\right) \times \|f\|_\infty \\ &= \frac{(Ct^{\alpha/2})^{N+1}}{\Gamma((N+1)\alpha/2+1)}\|f\|_\infty \to 0,\end{aligned}$$

as $N \to \infty$. Thus the proof is complete.

4. Proof of Theorem 2.4

Recall that $p_t^n(x;x')$, $q_t^n(x,\ell;x',\ell')$ and $p_t^n(x,\ell;x',\ell')$ are defined in Subsection 2.3 and M_μ is the constant introduced in Proposition 2.7.

4.1. An upper estimate for $p_t(x;x')$.

Lemma 4.1. *For any $\mu > 1$, and for any $n \in \mathbb{N}$, the inequality*

$$|p_t^n(x;x')| \le C_0' \frac{(M' t^{\alpha/2})^n}{\Gamma(n\alpha/2+1)} \tilde{H}_0(x-x';\mu\bar{a}t), \qquad (t,x,x') \in (0,T] \times \bar{D}^2,$$

holds true. Here we put $C_0' = M_\mu$ and $M' = M_\mu \Gamma(\alpha/2)$.

Proof. From (2.5), $|p_t^0(x;x')| \le M_\mu \tilde{H}_0(x-x',\mu\bar{a}t)$. Thus the assertion for $n=0$ holds. Suppose that the assertion holds true for n. Then from the hypothesis of induction, (2.7), (2.9), and (1.4), we have

$$\begin{aligned}|p_t^{n+1}(x;x')| &\le C_0' \frac{(M')^n\, M_\mu}{\Gamma(n\alpha/2+1)} \int_0^t s^{n\alpha/2}\,(t-s)^{\alpha/2-1}\, ds\ \tilde{H}(x-x',\mu\bar{a}t) \\ &\le C_0' \frac{(M' t^{\alpha/2})^{n+1}}{\Gamma((n+1)\alpha/2+1)} \tilde{H}_0(x-x',\mu\bar{a}t).\end{aligned}$$

Therefore, by induction, the assertion holds for all $n \in \mathbb{N}$. □

4.2. An upper estimate for $q_t^n(x,\ell;x',\ell')$.

Let us start with a technical lemma.

Lemma 4.2. *There exits a positive constant $C(\alpha)$ depending only on $\alpha \in (1/2,1)$ such that for each $n \in \mathbb{N}$*

$$\int_0^t s^{n\alpha/2}(t-s)^{(\alpha-2)/2} \left(1 \wedge \frac{x}{(t-s)^{1/2}}\right) ds \le C(\alpha)\, t^{(n+1)\alpha/2} \left(1 \wedge \left(\frac{x}{t^{1/2}}\right)^\alpha\right),$$

holds, for any $(t,x) \in (0,T] \times \bar{D}$.

Proof. Note that $B(n\alpha/2+1,\alpha/2) \le 2/\alpha$. Hence from (1.4) one can see

$$\int_0^t s^{n\alpha/2}\,(t-s)^{(\alpha-2)/2} \left(1 \wedge \frac{x}{(t-s)^{1/2}}\right) ds \le \frac{2}{\alpha} t^{(n+1)\alpha/2},$$

holds for any $(t,x) \in (0,T] \times \bar{D}$. If $x \geq t^{1/2}$, one can easily prove the assertion. Suppose next that $0 \leq x < t^{1/2}$. Then we have

$$\begin{aligned}
&\int_0^t s^{n\alpha/2} (t-s)^{(\alpha-2)/2} \left(1 \wedge \frac{x}{(t-s)^{1/2}}\right) ds \\
&= \int_0^{t-x^2} s^{n\alpha/2} (t-s)^{(\alpha-2)/2} \frac{x}{(t-s)^{1/2}} ds + \int_{t-x^2}^t s^{n\alpha/2} (t-s)^{(\alpha-2)/2} ds \\
&\leq x(t-x^2)^{n\alpha/2} \int_0^{t-x^2} (t-s)^{(\alpha-3)/2} ds + t^{n\alpha/2} \int_{t-x^2}^t (t-s)^{(\alpha-2)/2} ds \\
&\leq \left(\frac{2}{1-\alpha} + \frac{2}{\alpha}\right) x^\alpha t^{n\alpha/2}.
\end{aligned} \tag{4.1}$$

Thus we obtain the desired result with $C(\alpha) = \frac{2}{1-\alpha} + \frac{2}{\alpha}$. □

The following lemma is a modification of Lemma 4.1:

Lemma 4.3. *For any $\mu > 1$, there exists a positive constant C such that for any $n \in \mathbb{N}_0$ and for any $(t,x,x') \in (0,T] \times \bar{D}^2$,*

$$|p_t^n(x;x')| \leq C \frac{(M' t^{\alpha/2})^n}{\Gamma((n-1)\alpha/2+1)} \left(1 \wedge \left(\frac{x'}{t^{1/2}}\right)^\alpha\right) \tilde{H}_0(x-x', \mu\bar{a}t),$$

holds. Here $M' = M_\mu \Gamma(\alpha/2)$.

Proof. The assertion for $n=0$ follows from (2.5). By using Lemma 4.1, (2.7) and (2.9), we have

$$\begin{aligned}
|p_t^n(x,x')| &\leq \frac{C_0' M_\mu (M')^{n-1}}{\Gamma((n-1)\alpha/2+1)} \int_0^t s^{(n-1)\alpha/2}(t-s)^{(\alpha-2)/2} \left(1 \wedge \frac{x'}{(t-s)^{1/2}}\right) ds \\
&\quad \times \tilde{H}_0(x-x', \mu\bar{a}t) \\
&\leq \frac{C_0' C(\alpha)}{\Gamma(\alpha/2)} \cdot \frac{(M')^n t^{n\alpha/2}}{\Gamma((n-1)\alpha/2+1)} \left(1 \wedge \left(\frac{x'}{t^{1/2}}\right)^\alpha\right) \tilde{H}_0(x-x', \mu\bar{a}t).
\end{aligned}$$

Here in the last inequality we also use Lemma 4.2. Thus the proof is complete. □

Lemma 4.4. *For any $\mu > 1$, there exists a constant $\tilde{C}_0$ such that for any $n \in \mathbb{N}$ and for any $(t,x,\ell,x',\ell') \in (0,T] \times \bar{A}$,*

$$|q_t^n(x,\ell;x',\ell')| \leq \tilde{C}_0 \frac{(M')^{n-1} t^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)} h_t^\mu(x,\ell;x',\ell'), \tag{4.2}$$

holds. Here $M' = M_\mu \, \Gamma(\alpha/2)$.

Proof. Let us estimate the integrand of $q_t^n(x,\ell;x',\ell')$. We take a constant $1 < \mu' < \mu$. Using (2.8) with μ' and Lemma 4.3 we have

$$\begin{aligned}
|p_s^{n-1}(x;\xi)\, \kappa_{t-s}(\xi,\ell;x',\ell')| &\leq \frac{C M_{\mu'} \, (M')^{n-1}}{\Gamma((n-2)\alpha/2+1)} s^{(n-1)\alpha/2}(t-s)^{(\alpha-3)/2} \\
&\quad \times \left(\frac{\xi}{s^{1/2}}\right)^\alpha \tilde{H}_0(x-\xi, \mu\bar{a}s)\, h_{t-s}^{\mu'}(\xi,\ell;x',\ell').
\end{aligned} \tag{4.3}$$

Note that $\xi \le \xi + x' + \ell' - \ell$ for $(x,\ell,x',\ell') \in \bar{A}$. Hence by using Proposition 6.1 in Appendices, the right hand side above is dominated by

$$\begin{aligned}
&\frac{CM_{\mu'}\,(M')^{n-1}}{\Gamma((n-2)\alpha/2+1)}\,s^{(n-2)\alpha/2}(t-s)^{\alpha-3/2} \\
&\qquad \times \left(\frac{\xi + x' + \ell' - \ell}{(t-s)^{1/2}}\right)^{\alpha} \tilde{H}_0(x-\xi, \mu\bar{a}s) h^{\mu'}_{t-s}(\xi,\ell;x',\ell') \\
&\le \frac{CM_{\mu'}\,(M')^{n-1}}{\Gamma((n-2)\alpha/2+1)} s^{(n-2)\alpha/2}(t-s)^{\alpha-3/2}\tilde{H}_0(x-\xi,\mu\bar{a}s) h^{\mu}_{t-s}(\xi,\ell;x',\ell'),
\end{aligned}$$

Integrating both side with s and ξ, and using (2.10), and (1.4), we obtain (4.2). $\square$

4.3. An upper estimate for $p^n_t(x,\ell;x',\ell')$.

Lemma 4.5. *For any $\mu > 1$, there exist positive constants C''_0 and M'' such that for any $n \in \mathbb{N}_0$ and for any $(t,x,\ell,x',\ell') \in (0,T] \times \bar{A}$,*

$$|p^n_t(x,\ell;x',\ell')| \le C''_0 \frac{(M'')^n t^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)} h^{\mu}_t(x,\ell;x',\ell'),$$

holds true.

Proof. By induction with respect to n, we will prove Lemma 4.5 with constants

$$C''_0 = \max\{\tilde{C}_0, M_\mu\Gamma(1/2)\}, \qquad M'' = 3M'. \tag{4.4}$$

Here $\tilde{C}_0$ is the constant introduced in Lemma 4.4 and $M' = M_\mu\Gamma(\alpha/2)$. The assertion for $n = 0$ can be deduced from (2.6). Suppose that the assertion for n holds true. By using (2.7), (2.10) and (1.4) we have

$$\begin{aligned}
&\int_0^t s^{(n\alpha-1)/2}\,ds \int_D \left|h^{\mu}_s(x,\ell;\xi,\ell')\,\kappa_{t-s}(\xi;x')\right| d\xi \\
&\le \frac{M_\mu\Gamma((n\alpha+1)/2)\Gamma(\alpha/2)}{\Gamma((n+1)\alpha+1)/2)} t^{((n+1)\alpha-1)/2}\, h^{\mu}_t(x,\ell;x',\ell'),
\end{aligned}$$

and by using (2.8), (2.11) and (1.4)

$$\begin{aligned}
&\int_0^t s^{(n\alpha-1)/2}\,ds \int_\ell^{\ell'}\int_D \left|h^{\mu}_s(x,\ell;\xi,\lambda)\kappa_{t-s}(\xi,\lambda;x',\ell')\right| d\xi\,d\lambda \\
&\qquad \le \frac{M_\mu\,\Gamma((n\alpha+1)/2)\Gamma(\alpha/2)}{\Gamma((n+1)\alpha+1)/2)} t^{((n+1)\alpha-1)/2}\, h^{\mu}_t(x,\ell;x',\ell').
\end{aligned}$$

Thus, the hypothesis of induction yields

$$\begin{aligned}
&\int_0^t \left|S^*_{t-s}p^n_s(x,\ell;\cdot,\cdot)(x',\ell')\right| ds \\
&\le C''_0 \frac{(M'')^n}{\Gamma(((n+1)\alpha+1)/2)} (2M')\, t^{((n+1)\alpha-1)/2}\, h^{\mu}_t(x,\ell;x',\ell').
\end{aligned}$$

On the other hand, since $\tilde{C}_0 \le C''_0$ and $M' := M_\mu\Gamma(\alpha/2) \le M''$, Lemma 4.4 yields

$$|q^{n+1}_t(x,\ell;x',\ell')| \le C''_0 \frac{(M'')^n}{\Gamma(((n+1)\alpha+1)/2)} M' t^{((n+1)\alpha-1)/2}\, h^{\mu}_t(x,\ell;x',\ell').$$

Therefore by noting the definition of $p^{n+1}(x,\ell;x',\ell')$, one can show that the desired estimate for $n+1$ holds true. $\square$

4.4. Proof of Theorem 2.4. The upper estimates of $p_t(x;x')$ and $p_t(x,\ell;x',\ell')$ follow from Lemmas 4.1, and 4.5, respectively. It suffices to show that for any n and for any $f \in C_b(D\times\mathbb{R})$

$$I_t^n f(x,\ell) = \int_D f(x',\ell)p_t^n(x;x')\,dx' + \int_{\mathbb{R}}\int_D f(x',\ell')\,p_t^n(x,\ell;x',\ell')\,dx'\,d\ell', \quad (4.5)$$

holds true. We will prove this by induction. We first remark that $I_t^{n+1}f(x,\ell) = \int_0^t I_{u_1}^n(S_{t-u_1}f)(x,\ell)\,du_1$. Hence, we have by the hypothesis of induction that

$$\begin{aligned} &I_t^{n+1}f(x,\ell)\\ &= \int_0^t \left(\int_D S_{t-u_1}f(\xi,\ell)p_{u_1}^n(x;\xi)\,d\xi + \int_{\mathbb{R}}\int_D S_{t-u_1}f(\xi,\ell')\,p_{u_1}^n(x,\ell;\xi,\lambda)\,d\xi\,d\lambda\right)du_1. \end{aligned}$$

By using Fubini theorem, the first term in the right hand side is evaluated as

$$\begin{aligned} \int_0^t\int_D S_{t-u_1}f(\xi,\ell)p_{u_1}^n(x;\xi)\,d\xi\,du_1 &= \int_{\mathbb{R}}\int_D f(x',\ell')p_t^{n+1}(x;x')\,dx'\,\delta_\ell(d\ell')\\ &\quad + \int_{\mathbb{R}}\int_D f(x',\ell')q_t^{n+1}(x,\ell;x',\ell')\,dx'\,d\ell', \end{aligned}$$

and the second term is

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}}\int_D S_{t-u_1}f(\xi,\lambda)\,p_{u_1}^n(x,\ell;\xi,\lambda)\,d\xi\,d\lambda\,du_1\\ &= \int_{\mathbb{R}}\int_D f(x',\lambda)\left(\int_0^t\int_D p_{u_1}^n(x,\ell;\xi,\lambda)\,\kappa_{t-u_1}(\xi;x')\,d\xi\,du_1\right)dx'\,d\lambda\\ &\quad + \int_{\mathbb{R}}\int_D f(x',\ell')\left(\int_0^t\int_{\mathbb{R}}\int_D p_{u_1}^n(x,\ell;\xi,\lambda)\,\kappa_{t-u_1}(\xi,\lambda;x',\ell')\,d\xi\,d\lambda\,du_1\right)dx'\,d\ell'. \end{aligned}$$

Thus, noting the definition of $p_t^{n+1}(x,\ell;x',\ell')$, we have (4.5) for $n+1$.

5. Proof of Theorem 2.5

As we mentioned in Remark 2.6, we have $p_t^n(x,\ell;x',\ell') = p_t^n(x,0;x',\ell'-\ell)$ and $p_t(x,\ell;x',\ell') = p_t(x,0;x',\ell'-\ell)$. Hence, for the proof of Theorem 2.5, we may restrict our attention to the kernels $p_t^n(x,0,x';\ell')$ and $p_t(x,0,x';\ell')$ on

$$A'_\delta := \{(x,x',\ell')\in\mathbb{R}^3; x,x'\in D,\ \ell'>\delta\}.$$

Moreover we also note that $h_t^\mu(x,\ell;x',\ell') = h_t^\mu(x,0;x',\ell'-\ell)$ and $\kappa_t(x,\ell;x',\ell') = \kappa_t(x,0;x',\ell'-\ell)$. In this section, to simplify the notation, we put

$$\begin{aligned} p_t^n(x,x';\ell') &= p_t^n(x,0;x',\ell'), & p_t(x,x';\ell') &= p_t(x,0;x',\ell'),\\ h_t^\mu(x,x';\ell') &= h_t^\mu(x,0;x',\ell'), & \kappa_t(x,x';\ell') &= \kappa_t(x,0;x',\ell'). \end{aligned}$$

5.1. Some technical lemmas. Define

$$\zeta_t^n(x,x';\ell') := \int_0^t \int_D p_s^{n-1}(x,\xi;\ell')\kappa_{t-s}(\xi;x')\,d\xi\,ds,$$

$$\rho_t^n(x,x';\ell') := \int_0^t \int_0^{\delta/2} \int_D p_s^{n-1}(x,\xi;\lambda)\,\kappa_{t-s}(\xi,x',\ell'-\lambda)\,d\xi\,d\lambda\,ds, \tag{5.1}$$

$$\tilde{\rho}_t^n(x,x';\ell') := \int_0^t \int_{\delta/2}^{\ell'} \int_D p_s^{n-1}(x,\xi;\lambda)\,\kappa_{t-s}(\xi,x',\ell'-\lambda)\,d\xi\,d\lambda\,ds.$$

Then one can see that for any $(x,x',\ell') \in A'_\delta$

$$p_t^n(x,x';\ell') = q_t^n(x,x';\ell') + \zeta_t^n(x,x';\ell') + \rho_t^n(x,x';\ell') + \tilde{\rho}_t^n(x,x';\ell').$$

In Lemmas 5.1–5.4 below, we will show that the kernels $p_t^0(x,x';\ell')$, $q_t^n(x,x';\ell')$ and $\rho_t^n(x,x';\ell')$ are infinitely differentiable with respect to ℓ' and give also upper estimates on $\bar{A}'_\delta$ for them. Throughout this section, C_0 and M stand for the constants introduced in Theorem 2.4. We may assume that

$$\max\{1, M', M''\} \le M. \tag{5.2}$$

Here M' and M'' are constants introduced in Lemma 4.1 and Lemma 4.5, respectively. It should be emphasized that the constant C in Lemmas 5.1–5.4 below are independent of δ and n.

Lemma 5.1. *$p_t^0(x,x';\ell')$ is infinitely differentiable with respect to ℓ'. Moreover for any $m \in \mathbb{N}_0$ and for any $\mu > 1$ there exists a positive constant C such that for any $\delta \in (0,1)$ and for any $(t,x,x',\ell') \in (0,T] \times \bar{A}'_\delta$,*

$$\left|\partial_{\ell'}^m p_t^0(x,x';\ell')\right| \le \frac{C}{\delta^m} t^{-1/2} h_t^\mu(x,x';\ell'),$$

holds true. Here C depends on m, μ, $\bar{a}$, and $\underline{a}$.

Proof. We take a constant $1 < \mu' < \mu$. From Lemma 6.2 in Appendices, we have

$$\begin{aligned}\left|\partial_{\ell'}^m p_t^0(x,x';\ell')\right| &= \left|\tilde{H}_{m+1}(x+x'+\ell', a(z)t)\right| \\ &\le Ct^{-(m+1)/2}\tilde{H}_0(x+x'+\ell', \mu'\bar{a}t) \\ &\le \frac{C}{\delta^m} t^{-1/2} \left(\frac{x+x'+\ell'}{t^{1/2}}\right)^m h_t^{\mu'}(x,x';\ell').\end{aligned}$$

Here we used the fact that $x+x'+\ell' \ge \delta$ for $(x,x',\ell') \in \bar{A}'_\delta$. Thus the assertion follows from Proposition 6.1 in Appendices. □

Lemma 5.2. *The kernel $\kappa_t(x,x';\ell')$ is infinitely differentiable with respect to ℓ'. Moreover for any $m \in \mathbb{N}_0$ and for any $\mu > 1$ there exists a positive constant C such that for any $\delta \in (0,1)$ and for any $(t,x,x',\ell') \in (0,T] \times \bar{A}'_\delta$,*

$$\left|\partial_{\ell'}^m \kappa_t(x,x',\ell')\right| \le \frac{C}{\delta^m} t^{(\alpha-3)/2} h_t^\mu(x,x';\ell'),$$

holds true. Here C depends on m, μ, α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$ and T.

Proof. We take constants $1<\mu''<\mu'<\mu$. It follows from the Hölder continuity for a (**H2** of Assumption (H)) that

$$\begin{aligned}\left|\partial_{\ell'}^m \kappa_t(x,x';\ell')\right| &\le \|a\|_\alpha t^{\alpha/2}\left(\frac{|x-x'|}{t^{1/2}}\right)^\alpha \left|\tilde{H}_{m+3}(x+x'+\ell',a(x')t)\right| \\ &\quad + \|b\|_\infty \left|\tilde{H}_{m+2}(x+x'+\ell',a(x')t)\right| \\ &\le C\|a\|_\alpha t^{(\alpha-m-3)/2}\left(\frac{x+x'+\ell'}{t^{1/2}}\right)^\alpha h_t^{\mu''}(x,x';\ell') \\ &\quad + C\|b\|_\infty t^{-(m+2)/2} h_t^{\mu'}(x,x';\ell').\end{aligned}$$

Here we also use Lemma 6.2. Because $t^{-(m+2)/2}\le T^{(1-\alpha)/2}\times t^{(\alpha-m-3)/2}$, Proposition 6.1 shows that the right hand side above is dominated by

$$Ct^{(\alpha-m-3)/2}h_t^{\mu'}(x,x';\ell') \le \frac{C}{\delta^m}\, t^{(\alpha-3)/2}\left(\frac{x+x'+\ell'}{t^{1/2}}\right)^m h_t^{\mu'}(x,x';\ell').$$

Here we used the fact that $x+x'+\ell'\ge\delta$ for $(x,x',\ell')\in\bar{A}'_\delta$ again. Thus the assertion follows from Proposition 6.1. $\square$

Lemma 5.3. *For each $n\in\mathbb{N}$, $q_t^n(x,x';\ell')$ is infinitely differentiable with respect to ℓ'. Moreover for any $m\in\mathbb{N}_0$ and for any $\mu>1$ there exists a positive constant C such that for any n, for any $\delta\in(0,1)$ and for any $(t,x,x',\ell')\in(0,T]\times\bar{A}'_\delta$,*

$$|\partial_{\ell'}^m q_t^n(x,x';\ell')| \le \frac{C}{\delta^m}\cdot\frac{M^n t^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)} h_t^\mu(x,x';\ell'), \tag{5.3}$$

holds. Here C depends on m, μ, α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$ and T.

Proof. We take a constant $1<\mu'<\mu$. It follows from Lemma 4.3 with $\mu=\mu$ and Lemma 5.2 with $\mu=\mu'$ that

$$\begin{aligned}|p_s^{n-1}(x;\xi)\,\partial_{\ell'}^m\kappa_{t-s}(\xi,x';\ell')| &\le \frac{C}{\delta^m}\cdot\frac{(M')^{n-1}}{\Gamma((n-2)\alpha/2+1)} s^{(n-1)\alpha/2}(t-s)^{(\alpha-3)/2} \\ &\quad \times\left(\frac{\xi}{s^{1/2}}\right)^\alpha \tilde{H}_0(x-\xi,\mu\bar{a}s)h_{t-s}^{\mu'}(\xi,x';\ell').\end{aligned}$$

We remark that the right hand side above is the same as that of (4.3) up to constant multiple. Hence, by the same way as the proof of Lemma 4.4, we have

$$\begin{aligned}&|p_s^{n-1}(x;\xi)\,\partial_{\ell'}^m\kappa_{t-s}(\xi,x';\ell')| \\ &\le \frac{C}{\delta^m}\cdot\frac{M^n\, s^{(n-2)\alpha/2}(t-s)^{\alpha-3/2}}{\Gamma((n-2)\alpha/2+1)}\tilde{H}_0(x-\xi,\mu\bar{a}s)h_{t-s}^\mu(\xi,x';\ell').\end{aligned} \tag{5.4}$$

Here we also use (5.2). This yields in particular that

$$\begin{aligned}&\sup_{\ell'\in[\ell+\delta,+\infty)}|p_s^{n-1}(x;\xi)\,\partial_{\ell'}^m\kappa_{t-s}(\xi,x';\ell')| \\ &\le \frac{C}{\delta^m}\cdot\frac{M^n\, s^{(n-2)\alpha/2}(t-s)^{\alpha-3/2}}{\Gamma((n-2)\alpha/2+1)}\,\tilde{H}_0(x-\xi,\mu\bar{a}s)\tilde{H}_0(\xi+x',\mu\bar{a}(t-s)).\end{aligned}$$

The right hand side above belongs to $L^1((0,t)\times D,\, ds\,d\xi)$. Thus $q_t^n(x,x';\ell')$ is m times differentiable with respect to ℓ' on A'_δ and

$$\partial_{\ell'}^m q_t^n(x,x';\ell') = \int_0^t ds \int_D p_s^{n-1}(x;\xi)\,\partial_{\ell'}^m \kappa_{t-s}(\xi,x';\ell')d\xi,$$

holds. Using (5.4), (2.10) and (1.4) we obtain (5.3). □

Lemma 5.4. *For each $n\in\mathbb{N}$, $\rho_t^n(x,x';\ell')$ is infinitely differentiable with respect to ℓ'. Moreover for any $m\in\mathbb{N}$ and for any $\mu>1$ there exists a positive constant C such that for any n, for any $\delta\in(0,1)$ and for any $(t,x,x';\ell')\in(0,T]\times\bar{A}'_\delta$,*

$$\left|\partial_{\ell'}^m \rho_t^n(x,x';\ell')\right| \le \frac{C}{\delta^m}\cdot\frac{M^n t^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)}\,h_t^\mu(x,x';\ell'), \tag{5.5}$$

holds. Here C depends on m, μ, α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$ and T.

Proof. For $(x,x',\ell')\in\bar{A}'_\delta$, one can see $(\xi,x',\ell'-\lambda)\in\bar{A}'{}_{\delta/2}$ if $(\xi,\lambda)\in D\times(0,\delta/2)$. Hence it follows from Lemmas 4.5 and 5.2 that

$$\begin{aligned}&\left|\,p_s^{n-1}(x,\xi;\lambda)\,\partial_{\ell'}^m\kappa_{t-s}(\xi,x';\ell'-\lambda)\right|\\ &\le \frac{C}{(\delta/2)^m}\cdot\frac{M^{n-1}s^{((n-1)\alpha-1)/2}(t-s)^{\alpha/2-1}}{\Gamma(((n-1)\alpha+1)/2)}h_s^\mu(x,\xi;\lambda)h_{t-s}^\mu(\xi,x';\ell'-\lambda).\end{aligned} \tag{5.6}$$

This yields in particular that

$$\begin{aligned}&\sup_{\ell'\in[\delta,+\infty)}\left|\,p_s^{n-1}(x,\xi;\lambda)\,\partial_{\ell'}^m\kappa_{t-s}(\xi,x';\ell'-\lambda)\right|\\ &\le \frac{2^mC}{\delta^m}\cdot\frac{M^{n-1}s^{((n-1)\alpha-1)/2}(t-s)^{\alpha/2-1}}{\Gamma(((n-1)\alpha+1)/2)}\tilde{H}_0(x+\xi,\mu\bar{a}s)\,\tilde{H}_0(\xi+x',\mu\bar{a}(t-s)).\end{aligned}$$

The right hand side above belongs to $L^1((0,t)\times D\times(0,\delta/2), ds\,d\xi\,d\lambda)$. Therefore $\rho_t^n(x,x';\ell')$ is m times differentiable with respect to ℓ' and

$$\partial_{\ell'}^m\rho_t^n(x,x';\ell') = \int_0^t\int_0^{\delta/2}\int_D p_s^{n-1}(x,\xi;\lambda)\,\partial_{\ell'}^m\kappa_{t-s}(\xi,x';\ell'-\lambda)\,d\xi\,d\lambda\,ds,$$

holds. Next, we prove (5.5). We take a constant $1<\mu'<\mu$. By using Lemma 5.2 with $\mu=\mu'$ and (2.12), we have

$$\left|\partial_{\ell'}^m\kappa_{t-s}(\xi,x',\ell'-\lambda)\right| \le \frac{C}{\delta^m}(t-s)^{\alpha/2-1}h_t^\mu(\xi,x';\ell')\tilde{H}_0(\ell'-\lambda,\mu\bar{a}q(t-s)).$$

By using Lemma 4.5, one can prove (5.5) similarly to the proof of (2.11). □

5.2. Proof of Theorem 2.5. Let m_0 be an arbitrary natural number. For each n it is enough to show that:

Claim $\mathbf{A_n}$: $p_t^n(x,x';\ell')$ is m_0 times differentiable on $A' := \cup_{0<\delta<1}A'_\delta$ with respect to ℓ'. Moreover, for any $0\le m\le m_0$ there exist constants $\tilde{C}_m$ and $\tilde{M}_m$ such that for any $\delta\in(0,1)$ and for any $(x,x',\ell')\in\bar{A}'{}_\delta$,

$$\left|\partial_{\ell'}^m p_t^n(x,x';\ell')\right| \le \frac{\tilde{C}_m}{\delta^m}\cdot\frac{(\tilde{M}_m)^n t^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)}h_t^\mu(x,x';\ell'),$$

holds ture. Both constants $\tilde{C}_m$ and $\tilde{M}_m$ depend on m, μ, α, $\bar{a}$, $\underline{a}$, $\|b\|_\infty$ and T, but are independent of δ and n.

We first determine $\tilde{C}_m$ and $\tilde{M}_m$ $(0 \le m \le m_0)$. In view of Lemmas 5.1–5.4, for any $0 \le m \le m_0$, there exists a positive constant $\tilde{C}_m$ such that for any n, for any $\delta \in (0,1)$ and for any $(x, x', \ell') \in \bar{A}'_\delta$, the following inequalities hold true:

$$\begin{aligned} \left|\partial_{\ell'}^m p_t^0(x,x';\ell')\right| &\le \frac{\tilde{C}_m}{\delta^m} t^{-1/2} h_t^\mu(x,x';\ell'), \\ \left|\partial_{\ell'}^m \kappa_t(x,x';\ell')\right| &\le \frac{\tilde{C}_m}{\delta^m} t^{(\alpha-3)/2} h_t^\mu(x,x';\ell'), \end{aligned} \tag{5.7}$$

and

$$\left|\partial_{\ell'}^m q_t^n(x,x';\ell')\right| + \left|\partial_{\ell'}^m \rho_t^n(x,x';\ell')\right| \le \frac{\tilde{C}_m}{\delta^m} \cdot \frac{M^n t^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)} h_t^\mu(x,x';\ell'). \tag{5.8}$$

Here M is the constant introduced in Theorem 2.4. We may assume that $1 \le \tilde{C}_m \le \tilde{C}_{m+1}$ for any $0 \le m \le m_0 - 1$. Next, $\tilde{M}_m$ is defined recursively as follows

$$\begin{cases} \tilde{M}_0 = M, \\ \tilde{M}_m = \max\{\tilde{M}_{m-1},\, 2 + (m + (\pi\mu\bar{a}T/2)^{1/2}) \cdot 2^m \tilde{C}_m\}, \quad \text{for } 1 \le m \le m_0. \end{cases} \tag{5.9}$$

It should be remarked that $\tilde{C}_m$ and $\tilde{M}_m$ are independent of δ and n.

Now, by induction with respect to n, we will prove Claim A_n holds for any $n \in \mathbb{N}_0$. The case $n = 0$ has been already proved in Lemma 5.1. Suppose that Claim A_k holds for $0 \le k \le n$. Let us show that Claim A_{n+1} holds true. We have already shown that $q_t^{n+1}(x,x';\ell')$ and $\rho_t^{n+1}(x,x';\ell')$ are infinitely differentiable with respect to ℓ' and given the upper estimates for their derivatives (see Lemmas 5.3 and 5.4). Hence we restrict our attention to $\zeta_t^{n+1}(x,x';\ell')$ and $\tilde{\rho}_t^{n+1}(x,x';\ell')$ that are defined by (5.1).

Differentiability of $\zeta_t^{n+1}(x,x';\ell')$: For $(x,x',\ell') \in A'_\delta$ and for $\xi \in D$, by the hypothesis of induction and (2.7) we have

$$\begin{aligned} &|\partial_{\ell'}^m p_s^n(x,\xi;\ell')\kappa_{t-s}(\xi;x')| \\ &\quad \le \frac{\tilde{C}_m}{\delta^m} \frac{(\tilde{M}_m)^n s^{(n\alpha-1)/2}(t-s)^{\alpha/2-1}}{\Gamma((n\alpha+1)/2)}\, h_s^\mu(x,\xi;\ell')\tilde{H}_0(x'-\xi, \mu\bar{a}(t-s)). \end{aligned} \tag{5.10}$$

Thus we have

$$\begin{aligned} &\sup_{\ell' \in [\delta,+\infty)} |\partial_{\ell'}^m p_s^n(x,\xi;\ell')\kappa_{t-s}(\xi;x')| \\ &\quad \le \frac{\tilde{C}_m}{\delta^m} \cdot \frac{(\tilde{M}_m)^n (t-s)^{\alpha/2-1}}{\Gamma((n\alpha+1)/2)}\, \tilde{H}_0(x+\xi, \mu\bar{a}s)\tilde{H}_0(\xi - x', \mu\bar{a}(t-s)). \end{aligned}$$

The right hand side above belongs to $L^1((0,t) \times D, ds\, d\xi)$ and hence $\zeta_t^n(x,x',\ell')$ is m times differentiable on A'_δ with respect to ℓ' and

$$\partial_{\ell'}^m \zeta_t^{n+1}(x,x';\ell') = \int_0^t \int_D \partial_{\ell'}^m p_s^n(x,\xi;\ell')\kappa_{t-s}(\xi;x')\, d\xi\, ds,$$

holds. By using (5.10), (2.10) and (1.4), we have also

$$\left|\partial_{\ell'}^m \zeta_t^{n+1}(x,x';\ell')\right| \le \frac{\tilde{C}_m}{\delta^m} \cdot \frac{(\tilde{M}_m)^n t^{((n+1)\alpha-1)/2}}{\Gamma(((n+1)\alpha+1)/2)}\, h_t^\mu(x,x';\ell'). \tag{5.11}$$

Differentiability of $\tilde{\rho}_t^{n+1}(x, x'; \ell')$: We first remark that

$$\begin{aligned}
&\partial_{\ell'} \int_{\delta/2}^{\ell'} p_s^n(x, \xi; \lambda)\kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda \\
&= p_s^n(x, \xi; \ell')\, \kappa_{t-s}(\xi, x'; 0) + \int_{\delta/2}^{\ell'} p_s^n(x, \xi; \lambda)\partial_{\ell'}\kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda.
\end{aligned} \tag{5.12}$$

Note that $\partial_{\ell'}\kappa_t(x, x'; \ell' - \lambda) = -\partial_\lambda \kappa_t(x, x'; \ell' - \lambda)$. Hence by the integration by parts formula we can evaluate the second term of the right hand side of (5.12) as follows

$$\begin{aligned}
&\int_{\delta/2}^{\ell'} p_s^n(x, \xi; \lambda)\partial_{\ell'}\kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda \\
&= - p_s^n(x, \xi; \ell')\kappa_{t-s}(\xi, x'; 0) + p_s^n(x, \xi; \delta/2)\kappa_{t-s}(\xi, x'; \ell' - \delta/2) \\
&\quad + \int_{\delta/2}^{\ell'} \partial_\lambda p_s^n(x, \xi; \lambda)\, \kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda.
\end{aligned}$$

Therefore the first term of the right hand side of (5.12) is canceled, and we obtain

$$\begin{aligned}
&\partial_{\ell'} \int_{\delta/2}^{\ell'} \kappa_{t-s}(\xi, x', \ell' - \lambda)\, p_s^n(x, \xi; \lambda)\, d\lambda \\
&= p_s^n\big(x, \xi; \delta/2\big)\, \kappa_{t-s}\big(\xi, x'; \ell' - \delta/2\big) + \int_{\delta/2}^{\ell'} \partial_\lambda p_s^n(x, \xi; \lambda)\kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda.
\end{aligned}$$

Iterating this procedure m times we have

$$\begin{aligned}
&\partial_{\ell'}^m \int_{\delta/2}^{\ell'} p_s^n(x, \xi; \lambda)\kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda \\
&= \sum_{j=0}^{m-1} \partial_{\ell'}^j p_s^n\big(x, \xi; \delta/2\big)\, \partial_{\ell'}^{m-1-j}\kappa_{t-s}\big(\xi, x'; \ell' - \delta/2\big) \\
&\quad + \int_{\delta/2}^{\ell'} \partial_{\ell'}^m p_s^n(x, \xi; \lambda)\kappa_{t-s}(\xi, x'; \ell' - \lambda)\, d\lambda.
\end{aligned}$$

Since $(x, x', \ell') \in A'_\delta$, one can see $(x, \xi, \delta/2)$, $(\xi, x', \ell' - \delta/2) \in \bar{A}'_{\delta/2}$ whenever $\xi \in D$. Therefore by the hypothesis of induction and (5.7) we have

$$\begin{aligned}
&\left|\partial_{\ell'}^j p_s^n(x, \xi; \delta/2)\, \partial_{\ell'}^{m-1-j}\kappa_{t-s}(\xi, x'; \ell' - \delta/2)\right| \\
&\leq \frac{\tilde{C}_j}{(\delta/2)^j} \cdot \frac{\tilde{C}_{m-1-j}}{(\delta/2)^{m-1-j}} \cdot \frac{(\tilde{M}_j)^n s^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)}\, (t-s)^{\alpha/2-1} \\
&\quad \times h_s^\mu(x, \xi; \delta/2)\, h_{t-s}^\mu(\xi, x'; \ell' - \delta/2) \\
&\leq \frac{2^m \tilde{C}_m^2 \tilde{M}_m^n s^{(n\alpha-1)/2}(t-s)^{\alpha/2-1}}{\delta^m \Gamma((n\alpha+1)/2)}\, h_s^\mu(x, \xi; \delta/2) \cdot h_{t-s}^\mu(\xi, x'; \ell' - \delta/2).
\end{aligned}$$

Here in the last inequality we use $\tilde{C}_j \le \tilde{C}_m$, $\tilde{M}_j \le \tilde{M}_m$. Note that $(x,\xi,\lambda) \in \bar{A}'_{\delta/2}$ if $(\xi,\lambda) \in D \times (\delta/2, \ell')$. Hence the hypothesis of induction and (2.8) also yields

$$\begin{aligned}
&\int_{\delta/2}^{\ell'} \left|\partial_{\ell'}^m p_s^n(x,\xi;\lambda)\kappa_{t-s}(\xi,x';\ell'-\lambda)\right| d\lambda \\
&\quad \le \frac{\tilde{C}_m}{(\delta/2)^m} \cdot \frac{(\tilde{M}_m)^n s^{(n\alpha-1)/2}(t-s)^{\alpha/2-1}}{\Gamma((n\alpha+1)/2)} \int_{\delta/2}^{\ell'} h_s^\mu(x,\xi;\lambda) h_{t-s}^\mu(\xi,x';\ell'-\lambda)\, d\lambda \\
&\quad \le \frac{2^m(\tilde{C}_m)^2}{\delta^m} \cdot \frac{(\tilde{M}_m)^n s^{(n\alpha-1)/2}(t-s)^{\alpha/2-1}}{\Gamma((n\alpha+1)/2)} \exp\left\{-(2\xi)^2/2\mu\bar{a}t\right\} \cdot h_t^\mu(x,x';\ell').
\end{aligned}$$

Here we used the fact that $\tilde{C}_m \ge 1$ and the semigroup property of $h_t^\mu(x,x';\ell') = \tilde{H}_0(x+x'+\ell';\mu\bar{a}t)$. Therefore we obtain

$$\begin{aligned}
&\left|\partial_{\ell'}^m \int_{\delta/2}^{\ell'} p_s^n(x,\xi;\lambda)\kappa_{t-s}(\xi,x';\ell'-\lambda)\, d\lambda\right| \\
&\le \frac{\tilde{C}_m}{\delta^m} \cdot \frac{2^m \tilde{C}_m\, (\tilde{M}_m)^n s^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)} (t-s)^{\alpha/2-1} \\
&\quad \times \left\{ m\, h_s^\mu\big(x,\xi;\delta/2\big)\, h_{t-s}^\mu(\xi,x';\ell'-\delta/2) + e^{-(2\xi)^2/2\mu\bar{a}t} \cdot h_t^\mu(x,x';\ell')\right\}.
\end{aligned} \tag{5.13}$$

In particular we have

$$\begin{aligned}
&\sup_{\ell' \in [\delta,+\infty)} \left|\partial_{\ell'}^m \int_{\delta/2}^{\ell'} p_s^n(x,\xi;\lambda)\kappa_{t-s}(\xi,x';\ell'-\lambda)\, d\lambda\right| \\
&\le \frac{\tilde{C}_m}{\delta^m} \cdot \frac{2^m \tilde{C}_m (\tilde{M}_m)^n s^{(n\alpha-1)/2}}{\Gamma((n\alpha+1)/2)} (t-s)^{\alpha/2-1} \\
&\quad \times \left\{ m \cdot \tilde{H}_0(x+\xi,\mu\bar{a}s)\tilde{H}_0(\xi+x',\mu\bar{a}(t-s)) + e^{-(2\xi)^2/2\mu\bar{a}t}\, \tilde{H}_0(x+x',\mu\bar{a}t)\right\}.
\end{aligned}$$

The right hand side above belongs to $L^1((0,t)\times D, ds\, d\xi)$ and hence $\tilde{\rho}_t^n(x,x';\ell')$ is m times differentiable and

$$\partial_{\ell'}^m \tilde{\rho}_t^n(x,x';\ell') = \int_0^t \int_D \left(\partial_{\ell'}^m \int_{\delta/2}^{\ell'} p_s^n(x,\xi;\lambda)\kappa_{t-s}(\xi,x';\ell'-\lambda)\, d\lambda \right) d\xi\, ds,$$

holds. By using (5.13), we have

$$\begin{aligned}
&\left|\partial_{\ell'}^m \tilde{\rho}_t^n(x,x';\ell')\right| \\
&\le \frac{\tilde{C}_m}{\delta^m} \cdot \frac{(m+(\pi\mu\bar{a}T/2)^{1/2}) \times 2^m \tilde{C}_m (\tilde{M}_m)^n t^{((n+1)\alpha-1)/2}}{\Gamma(((n+1)\alpha+1)/2)} \times h_t^\mu(x,x';\ell').
\end{aligned} \tag{5.14}$$

The final step of the proof of Theorem 2.5: Now, one can conclude that $p_t^{n+1}(x,x',\ell')$ is m times differentiable with respect to ℓ' for arbitrary $m \le m_0$ on

A'_δ. Moreover by using (5.8), (5.11) and (5.14) we have

$$\begin{aligned}\left|\partial_{\ell'}^m p_t^{n+1}(x;x';\ell')\right| &\le |\partial_{\ell'}^m q_t^{n+1}(x,x';\ell')| + |\partial_{\ell'}^m \zeta_t^{n+1}(x,x';\ell')| \\ &\quad + |\partial_{\ell'}^m \rho^{n+1}(x,x';\ell')| + |\partial_{\ell'}^m \tilde{\rho}_t^{n+1}(x,x';\ell')| \\ &\le \frac{\tilde{C}_m}{\delta^m}\cdot\frac{(\tilde{M}_m)^{n+1}t^{((n+1)\alpha-1)/2}}{\Gamma((n+1)\alpha+1)/2)}h_t^\mu(x,x';\ell').\end{aligned}$$

Thus the proof is complete.

6. Appendices

6.1. Proof of Proposition 2.7. In this subsection we will prove Proposition 2.7.

Proposition 6.1. *Let p be a continuous function with polynomial growth. For any $\mu > 1$, there exists a positive constant C such that*

$$\left|p(x/\sqrt{a})\,\tilde{H}_0(x,a)\right| \le C\tilde{H}_0(x,\mu a),$$

holds true for any $x \in \mathbb{R}$ and $a > 0$. Here C depends on p, and μ.

Proof. Let $\mu' > 1$ be a constant satisfying $1/\mu + 1/\mu' = 1$. We have

$$\left|p(x/\sqrt{a})\,\tilde{H}_0(x,a)\right| = \left|p(x/\sqrt{a})\right|\exp\left\{-\frac{x^2}{2\mu' a}\right\}\cdot\mu^{1/2}\,\tilde{H}_0(x,\mu a) \le C\tilde{H}_0(x,\mu a).$$

Here we set $C = \sup_z\left\{|p(z)|e^{-z^2/2\mu'}\right\}\times\mu^{1/2}$. □

Lemma 6.2. *For each $n \in \mathbb{N}_0$ and for any $\mu > 1$, there exists a positive constant C such that for any $z \in \bar{D}$ and for any $(t,x) \in (0,T)\times\bar{D}$*

$$\left|\tilde{H}_n(x,a(z)t)\right| \le Ct^{-n/2}\tilde{H}_0(x,\mu\bar{a}t), \tag{6.1}$$

holds. Here C depends on n, $\bar{a}$, $\underline{a}$ and μ.

Proof. For any $a > 0$, note that $\tilde{H}_0(x,a) = a^{-1/2}\tilde{H}_0(x/a^{1/2},1)$. Hence one obtains

$$\tilde{H}_n(x,a) = a^{-1/2}\frac{d^n}{dx^n}\tilde{H}_0(x/a^{1/2},1) = a^{-(n+1)/2}H_n(x/a^{1/2},1)\tilde{H}_0(x/a^{1/2},1).$$

It follows from Proposition 6.1 that for any $\mu > 1$ we can find a positive constant C' which depends on H_n and μ, and satisfies $\left|H_n(x/a^{1/2},1)\tilde{H}_0(x/a^{1/2},1)\right| \le C'\tilde{H}_0(x/a^{1/2},\mu) = C'a^{1/2}\tilde{H}_0(x,\mu a)$. Thus we obtain (6.1) after using **H1** of Assumption (H) . □

Proof of (2.5). We first note that for each $n = 0, 1, 2$ and for any $\mu > 1$, there exists a positive constant C such that

$$|\partial_x^n \bar{\pi}_t^{(z)}(x;x')| \le C\,t^{-n/2}\,\tilde{H}_0(x-x',\mu\bar{a}t), \tag{6.2}$$

holds for any $t \in (0,T]$ and for any $(x,x') \in \bar{D}\times\bar{D}$. Indeed we have by Lemma 6.2

$$\begin{aligned}|\partial_x^n \bar{\pi}_t^{(z)}(x,x')| &\le \left|\tilde{H}_n(x-x',a(z)t)\right| + \left|\tilde{H}_n(x+x',a(z)t)\right| \\ &\le C\,t^{-n/2}\left\{\tilde{H}_0(x-x',\mu\bar{a}t) + \tilde{H}_0(x+x',\mu\bar{a}t)\right\} \\ &\le C\,t^{-n/2}\tilde{H}_0(x-x',\mu\bar{a}t).\end{aligned}$$

In the last inequality we have used the fact that $\tilde{H}_0(x+x',\mu\bar{a}t) \le \tilde{H}_0(x-x',\mu\bar{a}t)$ for $(x,x') \in \bar{D}\times\bar{D}$.

Next, we suppose that $x \ge x' \ge 0$. Then Lemma 6.2 also shows that

$$\begin{aligned}|\partial_x^n \bar{\pi}_t^{(z)}(x;x')| &\le \int_{x-x'}^{x+x'} \left|\partial_y^{n+1}\tilde{H}_0(y,a(z)t)\right| dy \\ &\le Ct^{-(n+1)/2}\int_{x-x'}^{x+x'} \left|\tilde{H}_0(y,\mu\bar{a}t)\right| dy \\ &\le Ct^{-n/2}\frac{x'}{t^{1/2}}\tilde{H}_0(x-x',\mu\bar{a}t).\end{aligned} \tag{6.3}$$

Here we have used the fact that $\tilde{H}_0(y,\mu\bar{a}t) \le \tilde{H}_0(x-x',\mu\bar{a}t)$ for any $y \in [x-x',x+x']$, since we assumed that $x \ge x' \ge 0$. Therefore the desired inequality is valid for $x \ge x' \ge 0$.

Now suppose that $x' \ge x \ge 0$. For $n=0$, since $\pi_t^{(z)}(x;x') = \pi_t^{(z)}(x';x)$, one can show by the same procedure as (6.3) that

$$\left|\bar{\pi}_t^{(z)}(x;x')\right| \le C\cdot\frac{x}{t^{1/2}}\tilde{H}_0(x-x',\mu\bar{a}t) \le C\cdot\frac{x'}{t^{1/2}}\tilde{H}_0(x-x',\mu\bar{a}t), \tag{6.4}$$

because $x' \ge x \ge 0$. Thus we obtain the desired result for $n=0$.

We omit the proof of the case $n=1$, because it is similar to and simpler than that of the case $n=2$. Since $H_2(x,a) = -a^{-1}+(x/a)^2$, we have

$$\begin{aligned}\left|\partial_x^2\bar{\pi}_t^{(z)}(x;x')\right| \le \frac{1}{\underline{a}t}\left|\pi_t^{(z)}(x;x')\right| &+ \left(\frac{x-x'}{a(z)t}\right)^2\tilde{H}_0(x-x',\bar{a}(z)t) \\ &+ \left(\frac{x+x'}{a(z)t}\right)^2\tilde{H}_0(x+x',\bar{a}(z)t).\end{aligned}$$

By using (6.4), one can see

$$\frac{1}{\underline{a}t}\left|\pi_t^{(z)}(x;x')\right| \le \frac{C}{\underline{a}}t^{-1}\frac{x'}{t^{1/2}}\tilde{H}_0(x-x',\mu\bar{a}t).$$

From Proposition 6.1 we also have

$$\left(\frac{x-x'}{a(z)t}\right)^2\tilde{H}_0(x-x',a(z)t) \le \frac{C(x'-x)}{(a(z)t)^{3/2}}\tilde{H}_0(x-x',\mu\bar{a}t) \le \frac{Cx'}{(\underline{a}t)^{3/2}}\tilde{H}_0(x-x',\mu\bar{a}t),$$

and

$$\left(\frac{x+x'}{a(z)t}\right)^2\tilde{H}_0(x+x',\bar{a}(z)t) \le \frac{C(x'+x)}{(a(z)t)^{3/2}}\tilde{H}_0(x+x',\mu\bar{a}t) \le \frac{2Cx'}{(\underline{a}t)^{3/2}}\tilde{H}_0(x-x',\mu\bar{a}t),$$

because $x' \ge x \ge 0$. Thus the desired inequality for $n=2$ also holds true. □

Let us prove the rest of Proposition 2.7.

Proof of (2.6), (2.7), *and* (2.8) . By using Lemma 6.2 one can see (2.6). Let us prove (2.7). We take a constant $1 < \mu' < \mu$. By using (2.5) and Hölder continuity property of a, we have

$$\begin{aligned}|\kappa_t(x;x')| &\le C\left\{\|a\|_\alpha t^{\alpha/2-1}\left(\frac{|x-x'|}{t^{1/2}}\right)^\alpha\left(1\wedge\left(\frac{x'}{t^{1/2}}\right)\right)\tilde H_0(x-x',\mu'\bar a t)\right.\\&\qquad\left.+\|b\|_\infty t^{-1/2}\left(1\wedge\left(\frac{x'}{t^{1/2}}\right)\right)\tilde H_0(x-x',\mu\bar a t)\right\}\\&\le Ct^{\alpha/2-1}\left(1\wedge\frac{x'}{t^{1/2}}\right)\tilde H_0(x-x',\mu\bar a t).\end{aligned}$$

Here we have also used Proposition 6.1 and $t^{-1/2}\le T^{(1-\alpha)/2}\cdot t^{\alpha/2-1}$, in the last inequality. Thus we obtain (2.7). The proof of (2.8) follows the same line as the proof of Lemma 5.2. □

6.2. Proof of Lemma 3.1. Recall that $\mathcal{P}_t$ and S_t are defined in Subsection 2.3.

Lemma 6.3. *For any $f\in C_b(\bar D\times\mathbb{R})$, we have that $\lim_{t\to0}\mathcal{P}_tf(x,\ell)=f(x,\ell)$ holds for each $(x,\ell)\in\bar D\times\mathbb{R}$.*

Proof. Note that $\mathcal{P}_tf$ can be written as $\mathcal{P}_tf(x,\ell)=J_t^1f(x,\ell)+J_t^2f(x,\ell)$ with

$$\begin{aligned}J_t^1f(x,\ell)&=\int_0^\infty f(x',\ell)\bar\pi_t^{(x')}(x;x')\,dx',\\J_t^2f(x,\ell)&=\int_\ell^\infty\int_{\bar D}f(x',\ell')\bar\pi_t^{(x')}(x,\ell;x',\ell')\,dx'\,d\ell'.\end{aligned}\tag{6.5}$$

We first note that $J_t^1f(0,\ell)=0$ since $\pi_t^{(x')}(0,x')=0$ for any $x'\in\bar D$. We do the change of variable $(x',\ell')=(\sqrt{\bar at}\,\xi,\ell+\sqrt{\bar at}\,\lambda)=:(x(t,\xi),\ell(t,\lambda))$, and we see that $J_t^2f(0,\ell)$ is equal to

$$\int_0^\infty\int_0^\infty f(x(t,\xi),\ell+\ell(t,\xi))\frac{2\sqrt{\bar a}(\xi+\lambda)}{\sqrt{2\pi a(x(\xi,t))^3}}\exp\left\{-\frac{\bar a\cdot(\xi+\lambda)^2}{2a(x(\xi,t))}\right\}d\xi\,d\lambda.$$

Since f is bounded and a is uniformly elliptic, Lebesgue theorem shows that $J_t^2f(0,\ell)\to f(0,\ell)$, as $t\to0$. Hence we have $\mathcal{P}_tf(0,\ell)\to f(0,\ell)$, as $t\to0$. Next, suppose that $x\in D$. Then we do the change of variable $x'=x+\sqrt{\bar at}\,\xi=:x(t,\xi)$, and see that $J_t^1f(x,\ell)$ is equal to

$$\int_0^\infty f(x(t,\xi),\ell)\left(\tilde H_0(\sqrt{\bar at}\xi,a(x(t,\xi))t)-\tilde H_0(2x+\sqrt{\bar at}\xi,a(x(t,\xi))t)\right)\sqrt{\bar at}\,d\xi.$$

By the same reason as above, one can apply Lebesgue theorem again and we have $J_t^1f(x,\ell)\to f(x,\ell)$, as $t\to0$ for $(x,\ell)\in D\times\mathbb{R}$. On the other hand, it follows from (2.6) that

$$|\pi_t^{(x')}(x,\ell;x',\ell')|\le Ct^{-1/2}h_t^\mu(x,\ell;x',\ell')\le Ct^{-1/2}\exp\{-x^2/2\bar at\}\,\tilde H_0(x',\mu\bar at),$$

and hence

$$|J^2f(x,\ell)|\le C\|f\|_\infty t^{-1/2}\exp\{-x^2/2\bar at\}\to0,$$

as $t\to0$ since $x>0$. Therefore for $(x,\ell)\in D\times\mathbb{R}$, we have $\mathcal{P}_tf(x,\ell)\to f(x,\ell)$, as $t\to0$. Thus the proof is complete. □

Recall that $\mathcal{D}$ is defined by (2.2). We denote by $C_K^\infty(\bar{D}\times\mathbb{R})$ the space of smooth functions on $D\times\mathbb{R}$ with compact support.

Proposition 6.4. *For $f \in C_K^\infty(\bar{D}\times\mathbb{R})$, we have $\mathcal{P}_t f \in \mathcal{D}$.*

Proof. Since $f \in C_K^\infty(\bar{D}\times\mathbb{R})$, by applying Lebesgue theorem, one can see that $\partial_x(\mathcal{P}_t f)$, $\partial_x^2(\mathcal{P}_t f)$, and $\partial_\ell(\mathcal{P}_t f)$ exist. It remains to check that $\partial_\ell \mathcal{P}_t f(0,\ell) + \partial_x \mathcal{P}_t f(0,\ell) = 0$. Note first that

$$\begin{aligned}\partial_x \mathcal{P}_t f(x,\ell) = &\int_{\bar{D}} f(x',\ell)\partial_x \bar{\pi}_t^{(x')}(x;x')dx' \\ &+ \int_\ell^\infty \int_{\bar{D}} f(x',\ell')\partial_x \bar{\pi}_t^{(x')}(x,\ell;x',\ell')dx'\,d\ell'.\end{aligned}$$

Using the fact that $\bar{\pi}_t^{(x')}(0;x') = 0$ and the continuity of $f(x',\ell')\bar{\pi}_t^{(x')}(x,\ell;x',\ell')$ at $\ell'=\ell$, we also have

$$\begin{aligned}\partial_\ell \mathcal{P}_t f(0,\ell) = &-\int_{\bar{D}} f(x',\ell)\bar{\pi}_t^{(x')}(0,\ell;x',\ell)dx' \\ &+ \int_\ell^\infty \int_{\bar{D}} f(x',\ell')\partial_\ell \bar{\pi}_t^{(x')}(0,\ell;x',\ell')dx'\,d\ell'.\end{aligned}$$

Since for any $z\in\bar{D}$, $\partial_x \bar{\pi}_t^{(z)}(0;x') = -2\tilde{H}_1(x',a(z)t) = \bar{\pi}_t^{(z)}(0,\ell;x',\ell)$ and

$$\partial_\ell \bar{\pi}_t^{(z)}(0,\ell;x',\ell') + \partial_x \bar{\pi}_t^{(z)}(0,\ell;x',\ell') = 0,$$

one can see $\partial_\ell \mathcal{P}_t f(0,\ell) + \partial_x \mathcal{P}_t f(0,\ell) = 0$ holds. Thus the proof is complete. $\square$

Proposition 6.5. *If f is continuous function on $D\times\mathbb{R}$ with compact support, then* (3.1) *holds true.*

Proof. In view of Proposition 3.2 Hence it suffices to show that (3.1) holds true for any $f\in C_K^\infty(\bar{D}\times\mathbb{R})$. From Proposition 6.4, $\mathcal{P}_{t-u}f$ is in $\mathcal{D}$ if $f\in C_K^\infty(\bar{D}\times\mathbb{R})$. Thus Itô's formula shows

$$\begin{aligned}&E[\mathcal{P}_{t-s}f(X_s,\ell+L_s)] \\ &= \mathcal{P}_t f(x,\ell) + \int_0^s E[\partial_u \mathcal{P}_{t-u}f(X_u,\ell+L_u) + \mathcal{L}\mathcal{P}_{t-u}f(X_u,\ell+L_u)]du.\end{aligned}$$

Here $0<s<t$. We also have

$$\begin{aligned}&\partial_u \mathcal{P}_{t-u}f(x,\ell) \\ &= \int f(x',\ell)\partial_u \bar{\pi}_{t-u}^{(x')}(x;x')\,dx' + \int_{\mathbb{R}}\int_D f(x',\ell')\,\partial_u \bar{\pi}_{t-u}^{(x')}t(x,\ell;x',\ell')\,dx'\,d\ell' \\ &= -\int f(x',\ell)\bar{\mathcal{L}}^{(x')}\bar{\pi}_{t-u}^{(x')}(x;x')\,dx' - \int_{\mathbb{R}}\int_D f(x',\ell')\,\bar{\mathcal{L}}^{(x')}\bar{\pi}_{t-u}^{(x')}(x,\ell;x',\ell')\,dx'\,d\ell'.\end{aligned}$$

From here one obtains that

$$E[\mathcal{P}_{t-s}f(X_s,\ell+L_s)] - \mathcal{P}_t f(x,\ell) = \int_0^s E[S_{t-u}f(X_u,\ell+L_u)]du.$$

One can see that the right hand side above converges to $\int_0^t E[S_{t-u}f(X_u,\ell+L_u)]du$ as $s\to t$, in view of Proposition 3.2. On the other hand it follows from Lemma 6.3

that the left hand side above also converges to $E[f(X_t, \ell+L_t)] - \mathcal{P}_t f(x,\ell)$ as $s \to t$. Thus we conclude that (3.1) holds for any $f \in C_K^\infty(\bar{D} \times \mathbb{R})$. $\square$

Lemma 6.6. *Let $f \in C_b(D \times \mathbb{R})$. For $R > 0$, we consider a continuous function f_R such that $\|f_R\|_\infty \le \|f\|_\infty$ and*

$$f_R(x,l) = \begin{cases} f(x,l) & \text{if } x + |\ell| < R \\ 0 & \text{if } x + |\ell| > R+1, \end{cases}$$

holds. Let $K \subset D \times \mathbb{R}$ be an arbitrary compact set. Then inequalities hold:

(i) *For any $t \in (0,T]$ and for any large enough $R > 0$ we have*

$$\sup_{(x,\ell)\in K} |(\mathcal{P}_t f - \mathcal{P}_t f_R)(x,\ell)| \le C\|f\|_\infty \int_{\{\xi \in \mathbb{R}:\, |\xi| > R/8\bar{a}T\}} \tilde{H}_0(\xi,1)\, d\xi.$$

(ii) *For any $t \in (0,T]$ and for any large enough $R > 0$, we have*

$$\sup_{(x,\ell)\in K} |(S_t f - S_t f_R)(x,\ell)| \le C\|f\|_\infty\, t^{(\alpha-2)/2} \int_{\{\xi \in \mathbb{R}:\, |\xi| > R/8\bar{a}T\}} \tilde{H}_0(\xi,1)\, d\xi.$$

Proof. We prove (ii) only, since the proof of (i) is similar to and easier than that of (ii). We prove (ii) for any $R > 0$ satisfying $K \subset \{(x,\ell) \in \bar{D} \times \mathbb{R} :\, x + |\ell| \le R/2\}$. Using Proposition 3.2 and (3.2) we have

$$\begin{aligned}
&|(S_t f - S_t f_R)(x,\ell)| \\
&\le C\|f\|_\infty t^{(\alpha-2)/2} \Bigg\{ \int_{\{x' \in \bar{D}:\, x' + |\ell| > R\}} \tilde{H}_0(x - x', 2\bar{a}t)\, dx' \\
&\qquad + \int\int_{\{(x',\ell') \in \bar{D}\times\mathbb{R} :x' + |\ell'| > R\}} \tilde{H}_0(x - x'; 2\bar{a}t) \tilde{H}_0(\ell - \ell', 2\bar{a}t)\, dx'\, d\ell' \Bigg\} \\
&\le C\|f\|_\infty t^{(\alpha-2)/2} \Bigg\{ \int_{\{x' \in \mathbb{R}:\, |x-\xi| + |\ell| > R\}} \tilde{H}_0(\xi, 2\bar{a}t)\, d\xi \\
&\qquad + \int\int_{\{(\xi,\lambda) \in \mathbb{R}^2 :|x-\xi| + |\ell-\lambda| > R\}} \tilde{H}_0(\xi, 2\bar{a}t) \tilde{H}_0(\lambda, 2\bar{a}t)\, d\xi\, d\lambda \Bigg\}.
\end{aligned}$$

Here we also do the change of variable $\xi = x - x'$ in the first term and $(\xi,\lambda) = (x - \xi, \ell - \lambda)$ in the second term. Since $\{\xi \in \mathbb{R} :\, |\xi - x| + |\ell| > R\} \subset \{\xi \in \mathbb{R} :\, |\xi| > R/2\}$ for any $(x,\ell) \in K$, we have

$$\begin{aligned}
\int_{\{\xi \in \mathbb{R}:\, |\xi - x| + |\ell| > R\}} \tilde{H}_0(\xi, 2\bar{a}t)\, d\xi &\le \int_{\{\xi \in \mathbb{R}:\, |\xi| > R/2\}} \tilde{H}_0(\xi, 2\bar{a}t)\, d\xi \\
&\le \sqrt{T} \int_{\{\xi \in \mathbb{R}:\, |\xi| > R/4\bar{a}T\}} \tilde{H}_0(\xi, 1)\, d\xi.
\end{aligned}$$

We have also that for any $(x,\ell) \in K$

$$\begin{aligned}
&\{(\xi,\lambda) \in \mathbb{R}^2 \,:|x - \xi| + |\ell - \lambda| > R\} \\
&\subset \{(\xi,\lambda) \in \mathbb{R}^2 \,:|\xi| + |\lambda| > R/2\} \\
&\subset \{(\xi,\lambda) \in \mathbb{R}^2 \,:\, |\xi| > R/4\} \bigcup \{(\xi,\lambda) \in \mathbb{R}^2 \,:\, |\lambda| > R/4\},
\end{aligned}$$

and hence

$$\begin{aligned}
&\int\int_{\{(\xi,\ell')\in\mathbb{R}^2\,:|x-\xi|+|\ell-\lambda|>R\}} \tilde{H}_0(\xi,2\bar{a}t)\tilde{H}_0(\lambda,2\bar{a}t)\,d\xi\,d\lambda\\
&\le \int_{\{\xi\in\mathbb{R}\,:\,|\xi|>R/4\}} H_0(\xi,2\bar{a}t)\,d\xi + \int_{\{\lambda\in\mathbb{R}\,:\,|\lambda|>R/4\}} H_0(\lambda,2\bar{a}t)\,d\lambda\\
&\le 2\sqrt{T}\int_{\{\xi\in\mathbb{R}\colon|\xi|>R/8\bar{a}T\}} \tilde{H}_0(\xi,1)\,d\xi.
\end{aligned}$$

Therefore we obtain (ii). □

Proof of Lemma 3.1. For any $f \in C_b(D\times\mathbb{R})$, and for any $R>0$, we take f_R as in Lemma 6.6. Then one can see that

$$\big|P_Tf(x,\ell) - P_Tf_R(x,\ell)\big| \le 2\|f\|_\infty P(|X_T(x)| + |\ell + L_T(x)| \ge R),$$

which yields $\lim_{R\to\infty} P_Tf_R(x,\ell) = P_Tf(x,\ell)$. In view of (i) of Lemma 6.6 we also have $\lim_{R\to\infty}\mathcal{P}_tf_R(x,\ell) = \mathcal{P}_tf(x,\ell)$.

For any $R_0>0$, we set $K = \{(x,\ell) : x + |\ell| \le R_0\}$. Using Proposition 3.2 we have

$$\begin{aligned}
&|E[S_{T-u}f(X_u(x),\ell+L_u(x))] - E[S_{T-u}f_R(X_u(x),\ell+L_u(x))]|\\
&\le |E[S_{T-u}(f-f_R)(X_u(x),\ell+L_u(x)) :\, |X_u(x)|+|\ell+L_u(x)|\le R_0]|\\
&\qquad + |E[S_{T-u}(f-f_R)(X_u(x),\ell+L_u(x)) :\, |X_u(x)|+|\ell+L_u(x)|\ge R_0]|\\
&\le \sup_{(\xi,\lambda)\in K} |S_{T-u}(f-f_R)(\xi,\lambda)|\\
&\qquad + C(T-u)^{(\alpha-2)/2}\|f\|_\infty P(|X_u(x)|+|\ell+L_u(x)|>R_0).
\end{aligned}$$

Hence (ii) of Lemma 6.6 shows that

$$\begin{aligned}
&\limsup_{R\to\infty}\big|E[S_{T-u}f(X_u(x),\ell+L_u(x))] - E[S_{T-u}f_R(X_u(x),\ell+L_u(x))]\big|\\
&\le C(T-u)^{(\alpha-2)/2}\|f\|_\infty P(|X_u(x)|+|\ell+L_u(x)|>R_0),
\end{aligned}$$

holds. Because $R_0>0$ is arbitrary, this yields that

$$\lim_{R\to\infty}\int_0^T E[S_{T-u}f_R(X_u(x),\ell+L_u(x))]\,du = \int_0^T E[S_{T-u}f(X_u(x),\ell+L_u(x))]\,du.$$

From Proposition 6.5, (3.1) is valid for f_R. Therefore the approximation arguments above imply that (3.1) is also valid for any $f\in C_b(D\times\mathbb{R})$. □

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Masafumi Hayashi: University of the Ryukyus, Department of Mathematical Sciences, Faculty of Science, Nishihara-cho, Okinawa 903-0213, Japan

E-mail address: `hayashi@math.u-ryukyu.ac.jp`